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Noise-Induced Order II

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A curious noise effect in certain maps reported earlier is investigated further. A striking feature of these maps is obtained in the symbolic dynamical approach. The decrease of entropy is attributed to a simple mechanism which deletes certain states in the symbolic dynamics, and the value of the modified entropy is calculated based on this picture.

KEY WORDS: Map; symbolic dynamics; noise; entropy.

1. INTRODUCTION

A curious noise effect in a certain class of one-dimensional map is reported in Ref. 1. If we add noise to the chaotic dynamics generated from these maps, the entropy of the orbits decreases to 60%–70% of the original value at a critical noise amplitude. At the same time, the Lyapunov number decreases to a negative value and the spectrum shows a relatively sharp peak which is not observed originally. Moreover, the peaks in the invariant measure alter their positions as if the dynamics is roughly periodic. This paper is an attempt to understand this phenomenon.

Our approach is based on the symbolic dynamics. By this approach, we can clearly see the dissimilarity between the maps in the above class and the other maps such as the logistic model, and attribute the decrease of entropy to a very simple mechanism. In the presence of noise, the peculiarity of the former maps leads to the disappearance of certain symbolic dynamical states, and results in the decrease of entropy.

In Section 2, the definitions and tools for symbolic dynamical approach are presented. Some types of partition play a central role. In Section

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3 actual partitions and refinements (see Section 2 for definitions) are calculated on computer and explicitly show the peculiarity of the maps in the class mentioned above. Section 4 is devoted to the consideration of the noise effect from the viewpoint in this paper. The altered entropy value is predicted and compared with the value in the simulations for a certain map.

At present we do not have any simple criterion to say a given map is in the class above, but we can create one in the class by using an appropriate isomorphism. In Section 5 we show an example of map thus obtained. The last section contains summary and discussion.

2. SYMBOLIC DYNAMICS, REFINEMENT, AND ENTROPY

First, we introduce the notion of symbolic dynamics.

Consider a piecewise C^1 map f from unit interval $(0, 1)$ to itself. To such map we associate a minimal partition of unit interval such that on each element of it the map is monotonic and the number of the elements of the partition is minimum. In the logistic model, for example, the minimal partition A is given by $A = \{R, L\}$, where $R = (1/2, 1)$ and $L = (0, 1/2)$. If we associate a symbol to each element of the partition, we can associate a semi-infinite sequence of symbols to a point of unit interval. This correspondence is achieved when the nth symbol of the sequence is the symbol for the element of the partition which contains the nth iteration of that point by f.

In certain cases,[2] when the map has an absolutely continuous invariant measure, the correspondence between a point and a sequence of symbols is proved to be one-to-one with certain restrictions on the combinations of symbols. In the following cases, we assume that the correspondence is one-to-one and use freely the sequences of symbols in place of the orbits of f, without any claims of rigor.

To construct a symbolic dynamics out of f, other partitions than the minimal partition are possible. One possibility is the Markov partition.[3] This is a partition such that f is monotonic in every element of it and f maps the set of all the end points of the partition into itself. A map which has Markov partition is called Markov, but the existence of an absolutely continuous measure does not mean the map is Markov. But in many cases, a non-Markov map can be approximated as closely as desired by a Markov map. One of the advantages of this partition is that the rule to construct a sequence which has a corresponding point in unit interval is simple. In the following, we use the Markov partition frequently, while the minimal partition is convenient for computer simulations.

As a matter of notation, if we write a_i for a symbol of the symbolic dynamics, then we denote the corresponding element of the partition, (i.e., an interval in the unit interval), by (a_i) and call it "state a_i."

A partition A' is a refinement of A when each element of A' is contained in some element of A. We consider here a special refinement of partitions. A first refinement A_1 of the partition $A = \{(a_1), \ldots, (a_n)\}$ is defined as $A_1 = \{(a_1a_1), (a_1a_2), \ldots, (a_na_n)\}$, where $(a_ia_j) = (a_i) \cap f^{-1}((a_j))$ $(1 \leqslant i, j \leqslant n)$. Clearly (a_ia_j) are all disjoint. When $(a_i) \cap f^{-1}((a_j))$ is empty then we delete the (a_ia_j) from the list of A_1 above. In general we can define kth refinement of A, A_k, in the same way where its elements $(a_{i1} \ldots a_{ik+1})$ $= (a_{i1}) \cap f^{-1}((a_{i2})) \cap \cdots \cap f^{-k}((a_{ik+1}))$. Again in the case of logistic model, kth refinement is obtained when the set of points $\bigcup_{i=0}^{k}\{f^{-i}(0.5)\}$ is taken as a set of end points of the partition. An interval between two consecutive points represents a state of A_k.

Interpreting the symbolic sequences of a map as the products of an information source, we can calculate its Shannon entropy.[4] This is equal to the K–S entropy[5] of the absolutely continuous measure of the map if it exists. Let $p(a_{i1} \ldots a_{in})$ be the probability of the occurrence of the sequence $a_{i1} \ldots a_{in}$ in the symbolic dynamics, then by definition the entropy H is

$$H = -\operatorname*{limit}_{n\to\infty} n^{-1} \Sigma p(a_{i1} \ldots a_{in}) \ln(p(a_{i1} \ldots a_{in}))$$

where the summation is taken over all the possible sequences of symbols of length n. In this way, we can calculate the entropy in a computer simulation. But generally this expression converges very slowly. In computer simulations we rather assume that the symbolic dynamics is a Markov process of degree n, i.e., the $(n + 1)$th symbol depends only on the last n symbols, and use the following expression for the entropy H of a Markov process:

$$H = -\sum_{ij} p_i p_{ij} \ln(p_{ij})$$

where p_{ij} is the transition matrix of the Markov process and p_i is the stationary probability. The entropy obtained in this way is an upper bound for the real entropy of the original process.

An example of this way to calculate the entropy is shown in Fig. 1. This figure is obtained in the process of drawing Fig. 7, which shows the variation of entropy for a map as the noise level is varied. We take the minimal partition of the map and approximate the dynamics by Markov processes of various degree. Then, by the above expression, the entropy is calculated. In Fig. 1 the vertical axis is the entropy and the other two axes are the noise level and the degree of the Markov process used to approxi-

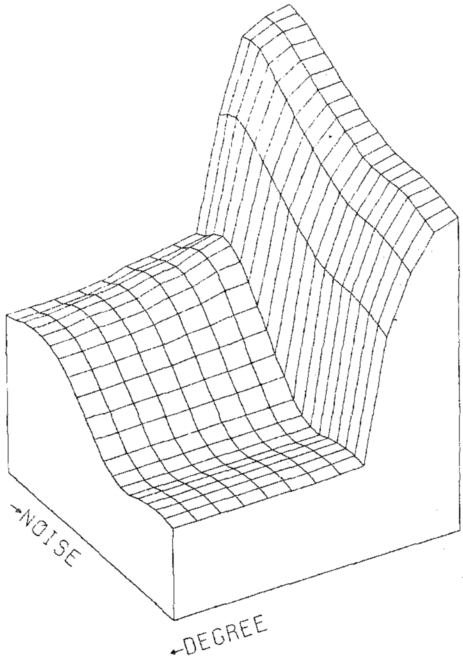


Fig. 1. Entropy plotted against the degree of Markov process and the noise level in the B–Z map with $b = 0.012137285937\ldots$.

mate the symbolic dynamics (degree 0 to 12). As a whole, the convergence of the values of entropy are good and the approximation by a Markov process of degree 12 is good enough for this purpose. Incidentally, the convergence is better when the noise level is larger.

3. NONUNIFORMITY IN THE WIDTH OF STATES

An element (a_i) of a partition $A = \{(a_1), \ldots, (a_n)\}$ has the width l_i in unit interval. We denote the width of an element $(a_{i1} \ldots a_{ik})$ of kth refinement A_k of A by $l_{i1 \ldots ik}$.

If the correspondence between the sequences of the symbols and points in unit interval is one-to-one, as assumed in Section 2, all the widths $l_{i1 \ldots ik}$ go zero, namely, the element $(a_{i1} \ldots a_{ik})$ shrinks to one point, as k goes infinity. But the manner of convergence can be different for different maps. Every element of A_k may have the same order of width or some elements may have very different order of width from the other elements. If

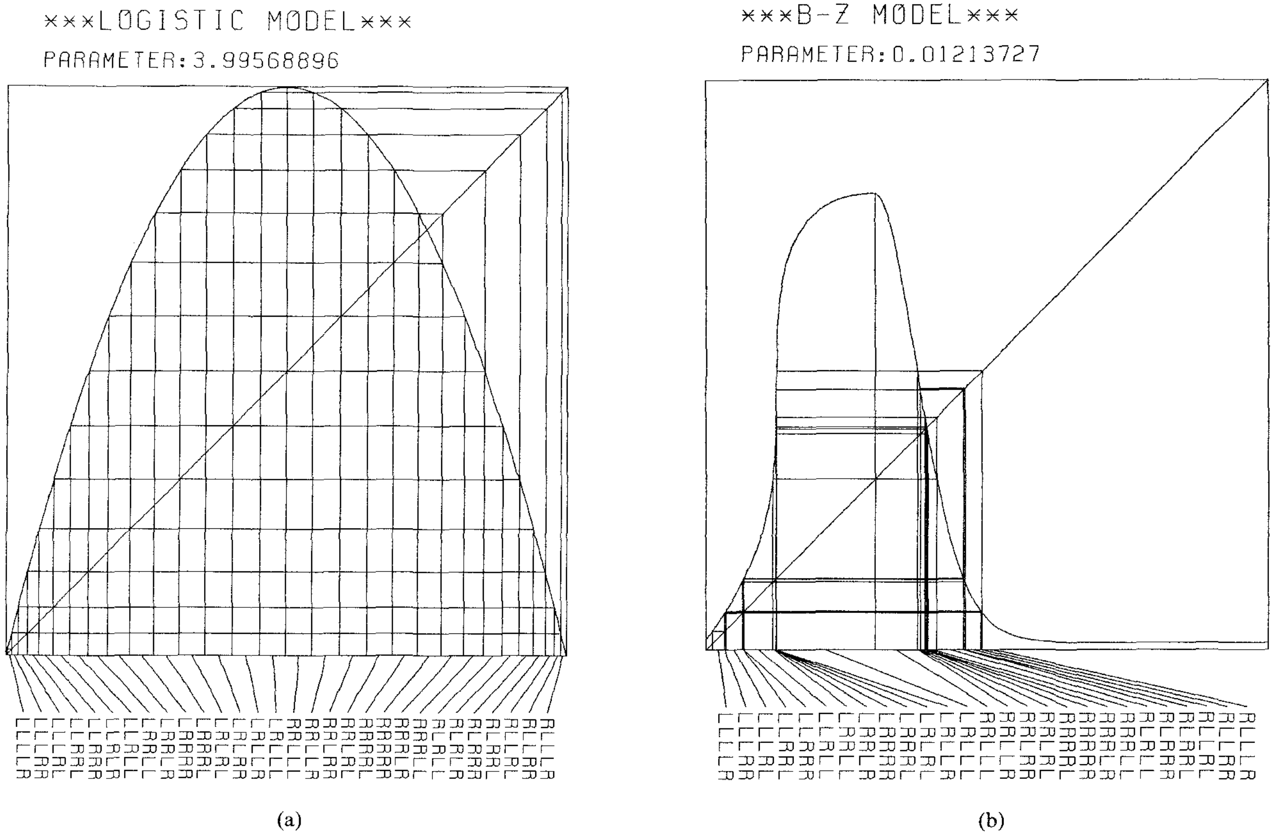


Fig. 2. Fourth refinement of a Markov partition in (a) logistic map and (b) B–Z map. In (a) $a = 3.9956936336\ldots$ and in (b) b is the same as in Fig. 1.

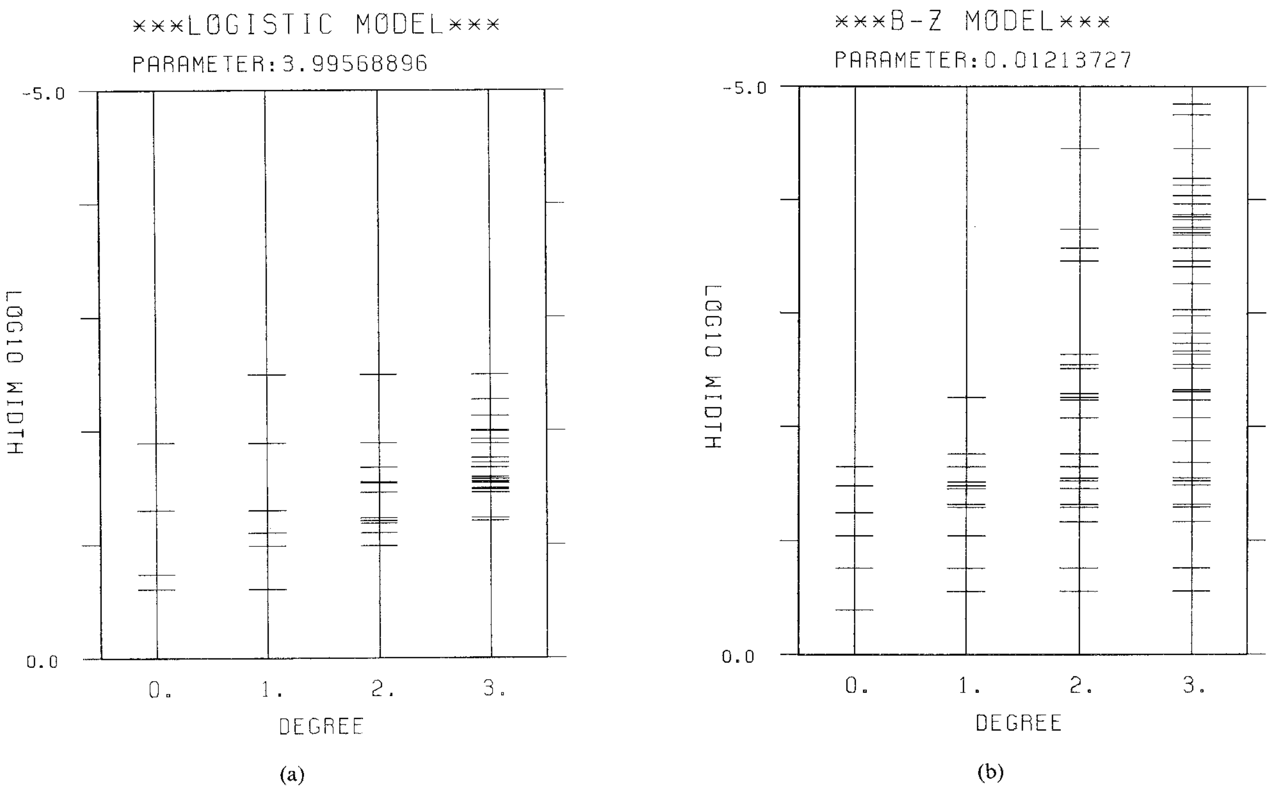


Fig. 3. Distribution of the width of states. The vertical axis is the width in log 10 scale. The figure under each vertical line is the degree of refinement of the Markov partition. The parameters are the same as in Fig. 2.

the former holds for every k in a map, the convergence of the states is uniform, and otherwise, the convergence is called nonuniform.

We now show two maps which display these two types of convergence of states. A map for the uniform convergence case is the logistic map,[6]

$$f(x) = ax(1 - x)$$

and a map for the nonuniform convergence case is the B–Z map,[7]

$$g(x) = \left[(x - 0.125)^{1/3} + 0.50607357\right]\exp(-x) + b, \qquad \text{for} \quad x < 0.3$$

$$g(x) = 0.121205692\left[10x\exp(-10/3x)\right]^{19} + b, \qquad \text{for} \quad x \geqslant 0.3$$

For each map, we take the minimal partition and show in Fig. 2 the fourth refinement of it. As stated in Section 2, in these maps the kth refinement is

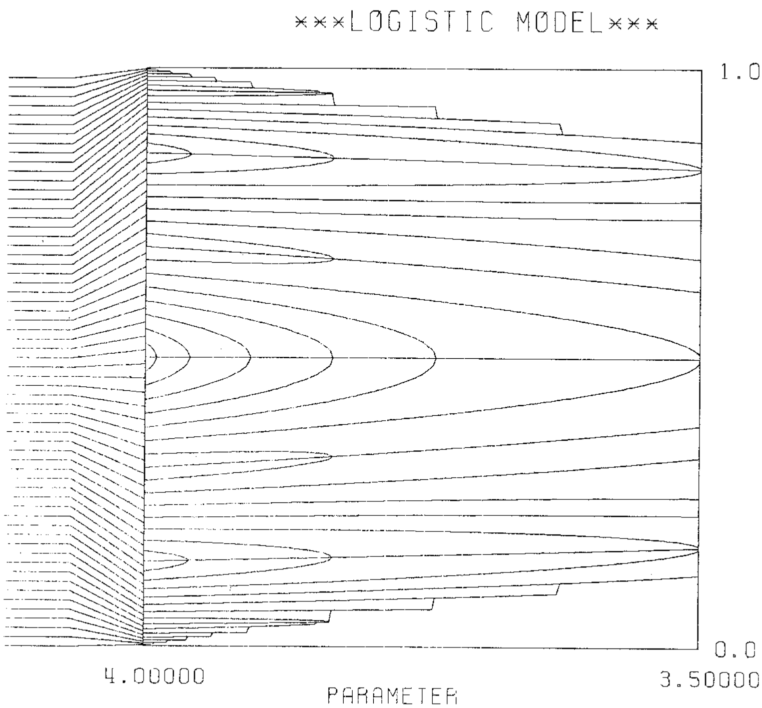


Fig. 4. Variation of fifth refinement of the minimal partition as the parameter is varied. The vertical axis is the unit interval and the horizontal axis is the parameter. (a) Logistic map.

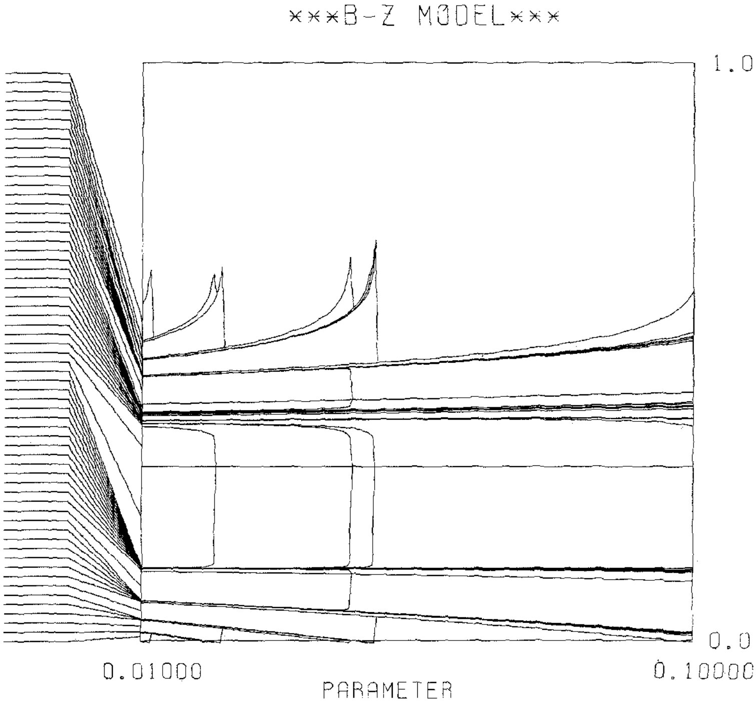


Fig. 4. (b) B–Z map.

obtained by taking the inverse images of the critical point (at which $df/dx = 0$) successively. In Fig. 2 the corresponding symbol for each state is indicated.

The difference is obvious. For the B–Z map the narrowest states are already out of resolution of this figure. To show the nonuniformity clearly, the widths of states are plotted in Fig. 3 for some refinements in the log 10 scale.

Incidentally, the maps presented each have a bifurcation parameter (a for logistic model and b for B–Z model). Then the question arises whether this type of convergence changes as the parameter is varied. Figure 4 shows the fifth refinement of minimal partitions for a range of the parameter in each map. As far as judged from this figure, the types of convergence of states do not change as the additive or multiplicative parameter is varied.

This nonuniformity in the width of states may lead to an interesting phenomenon when noise is added in the dynamics. The uniform noise in unit interval becomes very nonuniform when the dynamics is translated into symbolic dynamics.

4. APPROXIMATION TO NOISY DYNAMICS

What are the consequences of this nonuniform noise on the symbolic dynamics? A major one is to prevent the narrow states from occurring in the dynamics.

To see this, consider a map f on the unit interval and its partition $A = \{(a_1), \ldots, (a_n)\}$. Let l_i be the width of state a_i in the unit interval, and $p(a_i \mid a_j)$ the conditional probability of occurrence of state a_i after state a_j when the dynamics is without noise. Now add the rectangular noise of width $2d$ to the map f (Fig. 5a). We map a point x to $f(x)$ and according to the distribution law of noise, we disturb this point. By this process the conditional probability $p(a_i \mid a_j)$ will be altered. To estimate this, assume the uniform measure on interval (a_i) just mapped by f from state a_j (Fig. 5b).

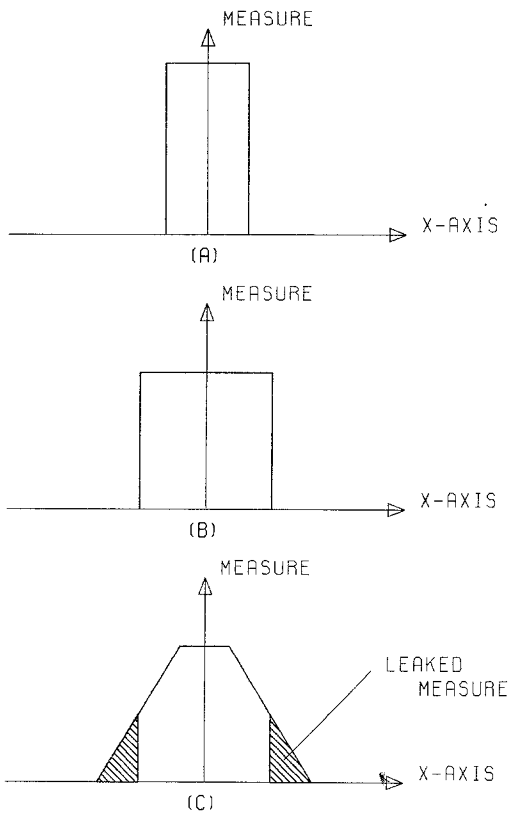


Fig. 5. (a) Rectangular noise. (b) The uniform measure on a state. (c) Disturbed measure of (b) by (a).

Then disturb this measure by the noise above. Then the leaked measure is $d/2l$ when $d < l$ and $1 - l/2d$ when $d \geqslant l$ (Fig. 5c). Suppose a_i is a narrow state and a_{i-1}, a_{i+1} are wide states, namely, $l_i \ll l_{i-1}, l_{i+1}$, state a_i being between the others in the unit interval. When noise is added to this situation the effects on the conditional probability $p(a_i \mid a_j)$ are (1) to decrease in $p(a_i \mid a_j)$ due to leaking, and (2) to increase in $p(a_i \mid a_j)$ due to leaking from $p(a_{i+1} \mid a_j)$, $p(a_{i-1} \mid a_j)$ and other transitions.

But if we set d (noise level) such that $l_i < d \ll l_{i-1}, l_{i+1}$ holds, we have large leaking from transition $a_j \rightarrow a_i$ and small leaking from $a_j \rightarrow a_{i-1}$ and $a_j \rightarrow a_{i+1}$. Therefore, unless $p(a_i \mid a_j)$ is considerably smaller than $p(a_j \mid a_{i-1})$ and $p(a_j \mid a_{i+1})$, the modified probability $p'(a_j \mid a_i)$ is substantially smaller than the original value. Since this argument is valid for any a_j, the occurrence of state a_i is suppressed by the noise.

We claim that this suppression of the narrow states is the sole reason for the decrease of entropy in the presence of noise. Let us estimate this value by modifying a dynamics so that the narrow states are deleted. For calculational convenience, we take a Markov process as the original dynamics.

To delete a state from a Markov process with transition matrix p_{ij} (conditional probability of transition $i \rightarrow j$) we must delete the corresponding row and column. We can delete immediately the row. But because of the condition that summation of p_{ij} over a row is 1, there is an arbitrariness in deleting the column. When the noise is symmetric and the leaks from the wide states are small, it is reasonable to divide evenly the probability on a narrow state among the nearest neighboring wide states. To take an example, if a narrow state a_i is between wide states a_{i+1} and a_{i-1}, the modified transition matrix $p'_{ji\pm 1}$ is given by $p_{ji\pm 1} + p_{ji}/2$. From this modified matrix p'_{ij} we calculate the modified stationary probability p'_i and its entropy. This modification generally results in the decrease of entropy.

We apply these methods to the B–Z map $b = 0.012137285937\ldots$. This is a Markov map. We can apply the above methods to any refinement of the Markov partition which has wide and narrow states. Of all these refinements, the largest change of entropy occurs for the smallest refinement. Therefore we consider the second refinement (Fig. 6). The narrow states are CAB, CAC, CAD, CAE. The appropriate noise is $d = 0.001$. We modified the matrix obtained by computer experiment without noise, and compare the modified entropy with simulation (Fig. 7). The entropy in the simulation is calculated based on the minimal partition, and a special "boundary condition." A word about this condition may be in order.

Noise in general has other effects than those discussed above. One of them is to kick the orbits out of the nonwandering set. This effect increases the entropy, and clearly it is not part of the above consideration. In the

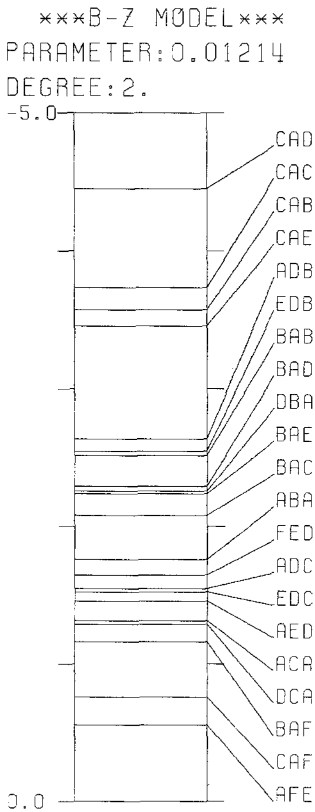


Fig. 6. Second refinement of a Markov partition of B–Z map with the same parameter as in Fig. 1. The vertical axis is the width of the states in log 10 scale. The symbol for each state is also indicated.

simulation in Fig. 7 we avoid this effect by taking a mirror boundary condition. This means that if a point is out of the nonwandering set, we take a mirror image of this point about the boundary and take this mirrored point as an orbital point. This amounts to modifying the distribution of noise near the boundary. In that figure the result of a simulation without this condition is also depicted as the free boundary case.

The agreement between the modified entropy and the simulation with the mirror boundary condition is good. The sudden decrease in entropy is indeed seen to occur at $d = 0.001$. The very good agreement in this case is due to the fact that the four narrow states we have deleted are isolated from the other states in Fig. 6. In such a situation our method of deleting the narrow states is expected to work very well in two points. First, we can

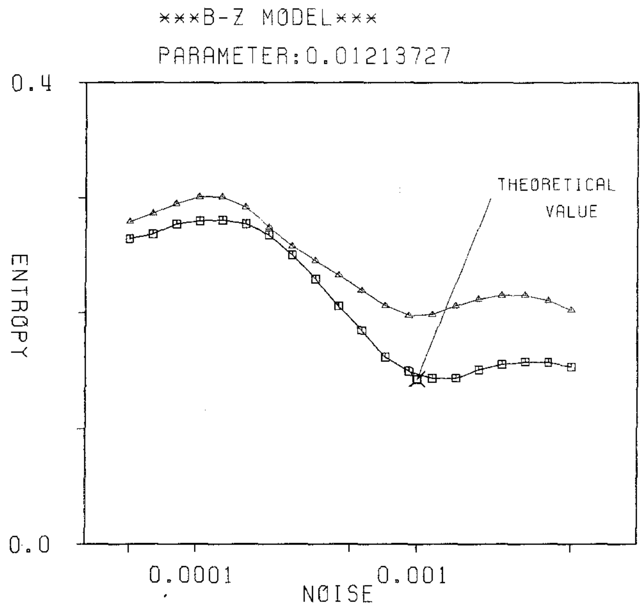


Fig. 7. Entropy of B–Z map $b = 0.0121372859\ldots$ (same as in Fig. 1) calculated based on the minimal partition. The symbolic dynamics is approximated by the Markov process of degree 12. See also Fig. 1. The noise is same as in Fig. 5a. The horizontal axis is the noise level in log 10 scale. Square, mirror boundary condition; triangle, free boundary condition. Theoretical value is also indicated.

select the narrow states unambiguously, and second, the noisy dynamics of the appropriate noise level is indeed described by the modified matrix. But in general maps with nonuniform states, these things are not expected to be satisfied, so our simple method may not lead to a clear-cut answer.

The difference of the entropy value between two boundary conditions is mainly due to the fact that, in the free boundary case, the sequence like *LLLLLR* occurs frequently, while this sequence seldom occurs in the mirror boundary case. In the latter case, sequences of L of length more than 4 occur very rarely. But this effect increases the entropy in the free boundary case gradually as the noise level is increased, and does not necessarily hinder the entropy-reducing mechanism. Actually, the sudden decrease of entropy occurs in both cases in a similar way. The mechanism for decreasing entropy appears to work in both cases in the same way.

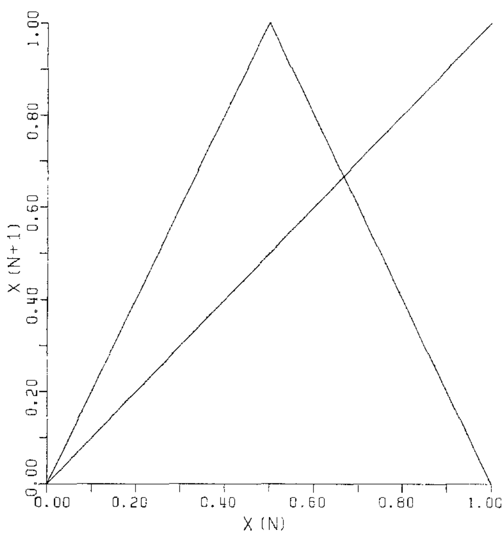
Moreover, the modified value of entropy persists for a range of the noise level. Namely, as the noise is further increased, other wider states begin to suffer from noise, (small variation in the entropy) but the four narrow states continue to be suppressed and this contributes large amount to the decrease of the entropy. Therefore, in this case, the four narrow states play a very important role in decreasing entropy.

Thus, the noisy dynamics is approximated by a dynamics which consists of the remaining states. The periodicity observed in the noisy B–Z model is attributed to periodicity in these remaining states.

5. THE ISOMORPHISM CONSIDERATION

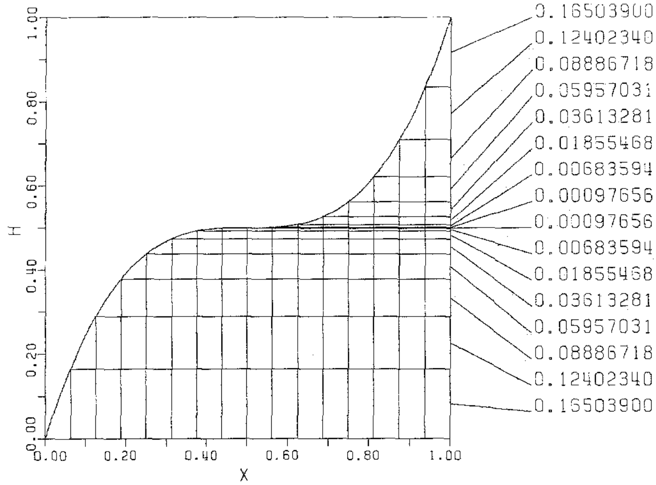
The property of nonuniform convergence in Section 3 is not invariant under isomorphism transformations. A map with uniform states can be transformed to a map with nonuniform states by an isomorphism between unit intervals. In this section we show briefly an example of this transformation.

We start with the tent map f (Fig. 8a). Its minimal partition and refinements are all uniform. The isomorphism h is Fig. 8b and $g = h \circ f \circ h^{-1}$ is shown in Fig. 8c. The minimal partition and its refinements of g are obtained by transforming that of f by h. The third refinements of the minimal partition of f and g are shown in Fig. 8b. From this, we can see that g has nonuniform states and the transition may occur near the noise level 0.001.

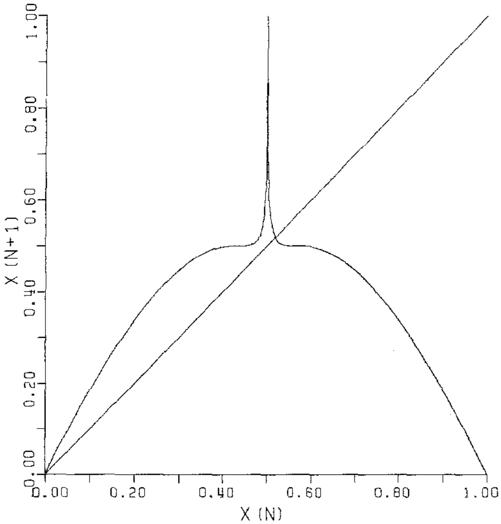


(a)

Fig. 8. (a) Tent map. (b) Isomorphism and transformation of a partition. The figures in the right-hand side are the width of the elements of transformed partition. (c) Transformation of tent map.

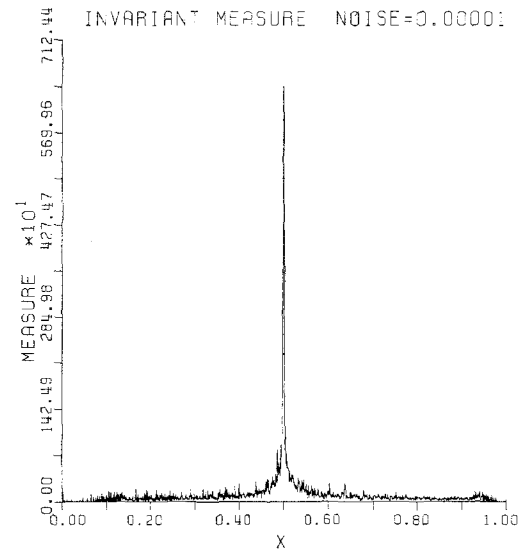


(b)

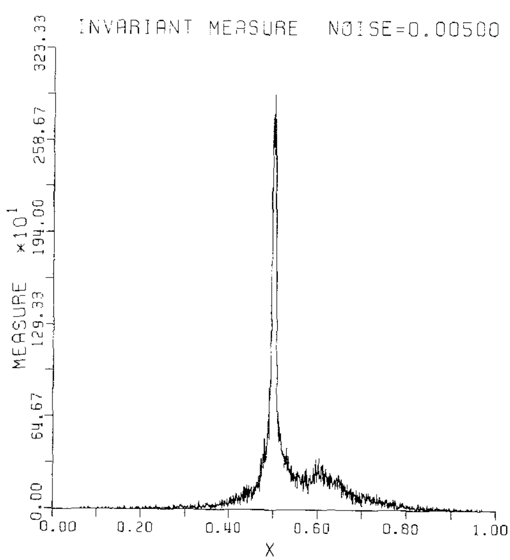


(c)

Fig. 8. Continued.



(a)



(b)

Fig. 9. Invariant measures of the map of Fig. 8c. Noise levels are (a) 0.00001 and (b) 0.005.

With g we calculate the invariant measures at the noise levels 0.00001 and 0.005 (Fig. 9a 9b). In the larger noise case we observe an extra peak which indicates the expected transition.

6. SUMMARY AND DISCUSSION

The scenario for the noise-induced order in this paper is as follows.

In certain maps, it is shown that the widths of the symbolic dynamical states are very nonuniform. In the presence of uniform noise on the map, the symbolic dynamics is suffered from very nonuniform noise because of the above nonuniformity. The major effect of this is to suppress the narrow states from the symbolic dynamics and this results in the sudden decrease in entropy at a critical noise level comparable to the width of the narrow states. We can calculate the entropy by approximating the noisy dynamics by a process lacking the narrow states.

Above the critical noise level, the dynamics consists of a few surviving states. In some cases, it happens that these surviving states have a very simple feature. For example, in the B–Z map a group of surviving states forms a nearly closed circle, i.e., transition probabilities between states are large for a sequence of transitions $a_1 \to a_2 \to \cdots \to a_1$. Thus the noisy dynamics appears to be periodic. This is the cause of the peak in spectrum observed in Ref. 1.

The map with nonuniform states is, therefore, considered to have a kind of "hidden dynamics" which consists only of the wide states. The noise-induced order is one instance of realization of this hidden dynamics. On the other hand, the hidden dynamics can be considered to be a description of the global features of the map. Therefore, it may reflect itself in other phenomena involving the map. For example, in the B–Z map, the hidden dynamics is nearly periodic for any parameter value, while in the bifurcation sequence of B–Z map simple stable periodic orbits share very large region in the parameter space. This may be considered as another instance of realization of the hidden dynamics.

In experimental situations, we frequently have a sufficient amount of noise to realize "the hidden dynamics." To describe these phenomena, by using "the hidden dynamics" in place of the rigorous dynamics is very interesting. We indeed see the possibility in the B–Z reaction. But these are left for a future study.

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